THE
MATHEMATICAL GAZETTE.

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THE MATHEMATICAL ASSOCIATION.

SYDNEY LOCAL BRANCH.

TAKING advantage of the presence in Sydney of the British Association for the Advancement of Science, the Local Branch of the Mathematical Association arranged a special meeting on August 24th to give the members an opportunity of hearing some of the distinguished visitors.

Although week-end excursions to places of interest in New South Wales had been arranged for the visitors, Professor Turner and Professor Perry remained in Sydney, and cheerfully gave their services to the Mathematical Association.

The gathering was large and representative. Professor Carslaw briefly introduced the speakers, and expressed for the members their appreciation of the high honour of hearing in Sydney two such well-known mathematicians from home.

Professor Turner was accorded a hearty reception. After giving some highly interesting historical references to the teaching of mathematics in the University of Cambridge and in the great Public Schools of England, he spoke on the work of the Mathematical Association and on the movement to improve the teaching of mathematics. He claimed that, where properly tried, the new method had been fairly successful. The difficulty was to get teachers to give the new method a fair trial. The old method of teaching was easier; the new method made more demands on the teachers.

After delighting his audience with his song:

"Here, a mere book, lies poor Euclides,
The darling of our schools,"

he spoke at length on the new method of teaching Geometry.

Professor John Perry gave a very interesting and highly instructive address, chiefly on the teaching of practical mathematics. Speaking of the teaching of Geometry, Professor Perry said that he took to Euclid as a duck takes to water, but that the mistake is made of thinking that all boys do the same. Whilst the beauty of Euclid appealed strongly to him, it did not strike the average boy that way. He had a great respect for the average boy. Carefully looked after, he gave good results. A great deal depended on the way a proposition was put before a student. How could you expect a student to argue logically on a topic he does not understand? A boy could not be taught whist by explaining the game to him, but put a pack of cards in his hand and he will learn the game in half an hour.

Professor Perry gave some very interesting details of his scheme of practical mathematics as taught to evening students in Technical Colleges in England.

Mr. J. T. Ewen, Inspector of Schools under the Scotch Education Department, and Professor Kerr Grant of Adelaide also spoke.

Before calling on Mr. A. H. Lucas of the Sydney Grammar School, and Mr. J. Nangle, Superintendent of Technical Instruction in New South Wales, to move and second a vote of thanks to speakers, Professor Carslaw remarked that the chief point in which he differed from Professor Perry was regarding the stage at which the practical method should give way to the demonstrational method in the teaching of Geometry. Everyone recognised the value of the practical method, especially as an introduction to geometrical ideas; but he thought there was a danger of deductive Geometry not receiving its proper place in the school work of the average boy.

Professor Turner and Professor Perry and our other visitors have placed the members of the local Association under a deep debt of gratitude for their instructive and inspiring addresses.

(Signed) R. J. MIDDLETON,
Hon. Sec. Sydney Local Branch.

KINETIC ENERGY, DIVIDED INTO MOLAR ENERGY AND AVAILABLE, INTERNAL, OR MOLECULAR ENERGY.

THE algebraical transformation

$$(1)\qquad W_1\frac{U_1^2}{2g}+W_2\frac{U_2^2}{2g}=\frac{(W_1U_1+W_2U_2)^2}{(W_1+W_2)2g}+\frac{W_1W_2}{W_1+W_2}\frac{(U_1-U_2)^2}{2g}$$

should be introduced into dynamical instruction at an early stage, and interpreted as the resolution or rearrangement of the left-hand side, the kinetic energy of two bodies, say railway trucks, of weight W_1 and W_2 tons, moving on the same line with velocity U_1 and U_2 feet per second, into (A) and (B) in the right-hand side.

The first term (A) may be called the *molar* energy, being the energy of the total weight, W_1+W_2 tons, moving with the velocity

$$(2)\qquad \frac{W_1U_1+W_2U_2}{W_1+W_2}$$

of the centre of gravity of the two bodies; while the second term (B) is the available energy, or the energy of the motion relative to the centre of gravity, as is seen on expressing it by

$$(3)\qquad W_1\left(U_1-\frac{W_1U_1+W_2U_2}{W_1+W_2}\right)^2\frac{1}{2g}+W_2\left(\frac{W_1U_1+W_2U_2}{W_1+W_2}-U_2\right)^2\frac{1}{2g}$$
$$=\frac{W_1W_2}{W_1+W_2}\frac{(U_1-U_2)^2}{2g}.$$

In any collision between the two bodies, which may take place either at the buffers of the two trucks, if the bodies are overtaking or approaching, but if receding may be supposed to act through a coupling chain becoming taut, the velocity of the centre of gravity, the total momentum, and the molar energy are unaltered.

Suppose W_1 overtakes W_2, and at the moment of collision the two trucks are coupled up; they move on together with the molar velocity of the centre of gravity, with the velocity of separation reduced to zero; the internal molecular energy $\frac{W_1W_2}{W_1+W_2}\frac{(U_1-U_2)^2}{2g}$ foot-tons has been dissipated,

or liberated, say in compressing the buffer springs; and the impact is called inelastic.

Conversely, with this amount of energy, acting explosively between the trucks when they are moving together, as with springs perfectly resilient, the original energy would be regained, and the bodies separate with their former velocity of approach; and denoting their velocity on separation by V_1 and V_2,

$$(4) \qquad V_2 - V_1 = U_1 - U_2, \text{ with } W_1V_1 + W_2V_2 = W_1U_1 + W_2U_2,$$

$$(5) \qquad U_1 + V_1 = U_2 + V_2 = 2\frac{W_1U_1 + W_2U_2}{W_1 + W_2}.$$

In this case of perfect resilience we can assume the Law of the Spring, and take it that with a feet total displacement of each buffer spring, and F the maximum buffer thrust in tons, the average thrust $\frac{1}{2}F$ tons acting through $2a$ feet requires Fa foot-tons of work, to be equated to the internal energy;

$$(6) \qquad Fa = \frac{W_1W_2}{W_1 + W_2}\frac{(U_1 - U_2)^2}{2g}.$$

During the impact against a spring resistance, the average velocity of approach is $\dfrac{U_1 - U_2}{\frac{1}{2}\pi}$, not $\frac{1}{2}(U_1 - U_2)$, as it would be against uniform resistance; and the impact lasts

$$(7) \qquad \frac{2a}{\dfrac{U_1 - U_2}{\frac{1}{2}\pi}} = \frac{\pi a}{U_1 - U_2} = \frac{\pi a}{\sqrt{(2ga)}} \cdot \frac{1}{\sqrt{\left(\dfrac{F}{W_1} + \dfrac{F}{W_2}\right)}} = \pi\sqrt{\frac{\frac{1}{2}a}{g}}\sqrt{\frac{1}{\dfrac{F}{W_1} + \dfrac{F}{W_2}}},$$

the beat of a pendulum of length

$$(8) \qquad \frac{\frac{1}{2}a}{\dfrac{F}{W_1} + \dfrac{F}{W_2}} = \frac{2W_1W_2}{W_1 + W_2} \cdot \frac{\frac{1}{4}a}{F},$$

where $\dfrac{2W_1W_2}{W_1 + W_2}$ is the H.M. of W_1 and W_2, and $\dfrac{a}{F}$ is constant, on the proportionality of a to F (Hooke's Law).

On Newton's experimental law of rebound, the velocity of separation after collision is a constant fraction e of the velocity of approach, where e is called the *coefficient* of *restitution*; then

$$(9) \qquad V_2 - V_1 = e(U_1 - U_2),$$

$$(10) \qquad V_1 + eU_1 = V_2 + eU_2 = (1+e)\frac{W_1U_1 + W_2U_2}{W_1 + W_2},$$

and the fraction e^2 of the internal energy has been dissipated in the impact; and

$$(11) \qquad \frac{W_1U_1 + W_2U_2}{W_1 + W_2} - V_1 = e\left(U_1 - \frac{W_1U_1 + W_2U_2}{W_1 + W_2}\right),$$

$$(12) \qquad V_2 - \frac{W_1U_1 + W_2U_2}{W_1 + W_2} = e\left(\frac{W_1U_1 + W_2U_2}{W_1 + W_2} - U_2\right),$$

so that in each body the velocity relative to the centre of gravity has been reduced by the factor e; and it implies that with spring buffers the resilient force of a spring is e^2 times the thrust required for the same compression; so that the rebound lasts longer, and the time in (7) must be divided by e to give its duration.

The extension to the case where the trucks are moving on lines which cross at an angle is made in *Matter and Motion*, lxxviii. lxxix. pp. 65, 66, by the use of the Vector method, and may well form an addition to Mr. Roberts' Note 432.

G. GREENHILL.

SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

An account of singular solutions of differential equations of the first order is given in every text-book on differential equations. The theory of singular solutions of differential equations of higher order is not usually given in elementary treatises. It is discussed by Goursat (*American Journal of Mathematics*, xi., 1889, pp. 329-372), who bases his discussion on the theory of singular solutions of simultaneous equations of the first order. From the didactic point of view a more elementary treatment may be acceptable, and an attempt is here made to indicate how singular solutions of a given equation of the second order may be found by simple means.

As in the case of equations of the first order we may deduce the singular solutions from the given differential equation $\phi(x, y, y_1, y_2)=0$, or from its complete primitive $f(x, y, a, b)=0$. (In this $y_1 \equiv \dfrac{dy}{dx}$, $y_2 \equiv \dfrac{d^2y}{dx^2}$, and a, b are arbitrary constants.)

It will be noted that a singular solution contains one arbitrary constant in general.

(I.) *A singular solution, if any, may be found as follows:*
Eliminate x *and* y *from*

$$\left.\begin{array}{r} f(x, y, a, b)=0, \quad \dfrac{\partial f}{\partial a}da+\dfrac{\partial f}{\partial b}db=0, \\ \left(\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial^2 f}{\partial y\,\partial a}-\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial^2 f}{\partial x\,\partial a}\right)da+\left(\dfrac{\partial f}{\partial x}\,\dfrac{\partial^2 f}{\partial y\,\partial b}-\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial^2 f}{\partial x\,\partial b}\right)db=0\,; \end{array}\right\}\text{.........(i)}$$

integrate the resulting differential equation of the first order in a *and* b, *thus expressing* b *as a function of* a *and an arbitrary constant; then the envelope of the singly infinite family of curves* $f(x, y, a, b)=0$, *now obtained, includes the singular solutions of* $\phi(x, y, y_1, y_2)=0$.

Suppose a singular solution represented by a curve c. Then through any point P of c an infinite number of curves of the family $f(x, y, a, b)=0$ passes. Choose b so that $f(x, y, a, b)=0$ touches c at P. If this is done for every point P of c, we express b as a function of a, and $f(x, y, a, b)=0$ becomes a singly infinite family whose envelope is c. Now at P, $f(x, y, a, b)=0$ and c have the same x, y, and y_1. But they both satisfy $\phi(x, y, y_1, y_2)=0$. Hence they have the same y_2 also, and therefore the singly infinite family $f(x, y, a, b)=0$ osculates its envelope at each point of contact.

Now, if $F(x, y, t)=0$ osculates its envelope at P, we readily prove that $F=0$ and $\dfrac{\partial F}{\partial t}=0$ touch at P, *i.e.*

$$F=0, \quad \frac{\partial F}{\partial t}=0, \quad \frac{\partial F}{\partial x}\cdot\frac{\partial^2 F}{\partial y\,\partial t}=\frac{\partial F}{\partial y}\cdot\frac{\partial^2 F}{\partial x\,\partial t}$$

at P. Applying this to $f(x, y, a, b)=0$, where b is a function of a, we have the result stated above.

A case of practical value is that in which $f(x, y, a, b)\equiv F(x+\xi, y+\eta, \zeta)$, where ξ, η, ζ are functions of a and b, and F is a homogeneous function of

$$x+\xi, \quad y+\eta, \quad \zeta.$$

The equation in a, b is then $F(d\xi, d\eta, d\zeta)=0$, (where, of course,

$$d\xi=\frac{\partial \xi}{\partial a}da+\frac{\partial \xi}{\partial b}db, \text{ etc.}).$$

For, if $F_1 \equiv \dfrac{\partial F}{\partial(x+\xi)}$, $F_{23} \equiv \dfrac{\partial^2 F}{\partial(y+\eta)\partial\zeta}$, etc., we have to eliminate x, y from

$$F=0, \quad F_1 d\xi + F_2 d\eta + F_3 d\zeta = 0,$$

and
$$F_1(F_{21}d\xi + F_{22}d\eta + F_{23}d\zeta) = F_2(F_{11}d\xi + F_{12}d\eta + F_{13}d\zeta);$$

and by Euler's theorem on homogeneous functions the last two equations give $(x+\xi):(y+\eta):\zeta = d\xi : d\eta : d\zeta$; whence $F(x+\xi, y+\eta, \zeta)=0$ gives

$$F(d\xi, d\eta, d\zeta)=0.$$

(II.) *A singular solution, if any, may be found as follows:*
Eliminate y_2 from

$$\phi(x, y, y_1, y_2)=0, \quad \frac{\partial\phi}{\partial y_2}=0. \qquad \text{(ii)}$$

Then the integral of the resulting differential equation of the first order includes the singular solution.

Suppose that in (I.) the curve c osculates two consecutive curves 1 and 2 of the singly infinite family $f(x, y, a, b)=0$ at Q and R.

Suppose that 1 and 2 meet at P, which is consecutive to Q and R. The curvature of 1 at P is in the limit equal to its curvature at Q, *i.e.* to the curvature of c at Q, *i.e.* in the limit to the curvature of c at R, *i.e.* to the curvature of 2 at R, *i.e.* in the limit to the curvature of 2 at P.

Suppose, to fix our ideas, that $\phi(x, y, y_1, y_2)=0$ is an algebraic equation of degree n in y_2. Then, in general, for given x, y, y_1 we get n different values for y_2. But we have just shown that if x, y, y_1 have the values given by the curve c at P, two of the n values of y_2 coincide; and hence $\dfrac{\partial\phi}{\partial y_2}=0$.

Of course, any solution found by the methods (I.) or (II.) must be tested to see whether it really satisfies the differential equation; just as in the case of differential equations of the first order.

For instance, in (II.) we get, besides the singular solution, the locus of a point on $f(x, y, a, b)=0$, at which two branches of this curve touch and have the same curvature there ("cusp of the second species"). The reader will find a discussion of this point in Goursat, *loc. cit.*

Ex. 1. $y_2^2 = 1 - y_1^2$ with complete primitive $y + \cos(x-a) - b = 0$.

Method I.—Equations (i) give, on eliminating x and y, $db = \pm da$, or $b = \pm a + K$. The required singular solution, if any, is therefore the envelope of $y + \cos(x-a) \mp a - K = 0$, *i.e.* $y = \pm x + k$; which is found on testing to be really a singular solution.

Method II.—Equations (ii) give $y_1 = \pm 1$, leading to $y = \pm x + k$, as before.

Ex. 2. $y_1 = y_2^2 + xy_2$ with complete primitive $y = ax^2 + 4a^2x + b$.

Method I.—Equations (i) give, on eliminating x and y, $db = 16a^2 da$, or $b = \frac{16}{3}a^3 + K$. The singular solution is the envelope of $y = ax^2 + 4a^2x + \frac{16}{3}a^3 + K$ or $y = -\frac{1}{12}x^3 + k$.

Method II.—Equations (ii) give, on eliminating y_2, $y_1 = -\frac{1}{4}x^2$ or

$$y = -\tfrac{1}{12}x^3 + k,$$

as before.

Ex. 3. Find the singular solutions, if any, of

$$cy_2^2 = (1+y_1^2)^3, \quad c^2y_2^2 + y_1^2 = 2yy_2,$$
$$y^2(yy_1 - xyy_2 + xy_1^2) = (yy_2 - y_1^2)^2.$$

Ex. 4. Find the singular solutions of the differential equation whose complete primitive is

$$x^2 + y^2 - 2a(2+3b^2)x + 4ab^3y - 3a^2b^4 = 0.$$

We may write this equation $(x+\xi)^2+(y+\eta)^2-\zeta^2=0$, where

$$\xi=-a(2+3b^2), \quad \eta=2ab^3, \quad \zeta=2a(1+b^2)^{\frac{3}{2}};$$

which is a homogeneous equation in $x+\xi$, $y+\eta$, ζ.

The required singular solution is therefore included in the envelope of the given circles when a and b are connected by the relation

$$(bda^2+4adadb)b^3=0,$$

$$d\xi^2+d\eta^2-d\zeta^2=0 \quad \text{or} \quad (b\,da^2+4a\,da\,db)b^3=0,$$

i.e. $a=$constant, or $ab^4=$constant, or $b=0$.

The envelope in the former case is $y^2=4ax$, as is geometrically evident, for the given circle osculates this parabola at $(ab^2, 2ab)$.

Ex. 5. Find the singular solutions of the differential equation whose complete primitive is

$$b^3(x^2+y^2)-(3b^4+1)ax-(b^4+3)ab^2y+3(b^4+1)a^2b=0,$$

$$xy-a\sin b(\sin^2 b+3\cos^2 b)x-a\cos b(3\sin^2 b+\cos^2 b)y+3a^2\sin b\cos b=0,$$

$$y^2+2a\cos^3 b\,x-2a\sin^3 b\,y+a^2(\sin^2 b-2\cos^2 b)=0.$$

The reader will readily apply similar methods to the finding of singular solutions of differential equations of higher orders. H. Hilton.

BERKELEY AND NEWTON.

In his interesting article on Newton in the July *Gazette* Dr. Rouse Ball had space only for a slight reference to Newton's views on what we should now call the Foundations of the Differential Calculus. The subject "involves," as Dr. Rouse Ball has remarked, "some awkward questions of philosophy which before Weierstrass's researches were usually slurred over." Some of the logical difficulties incident to Newton's method of approach were clearly seen and pointed out by a contemporary, the celebrated Bishop Berkeley, in a tract which drew much attention at the time, and was followed by a considerable controversy.

George Berkeley, who was born in 1685, when Newton was writing the *Principia*, was appointed to the bishopric of Cloyne in March, 1734, and the same month saw the publication of his tract *The Analyst*. Newton had died, full of years and honour, seven years before, and the long period of his complete dominance over British mathematicians had already begun. It is therefore not surprising that the *Analyst*, which attacked the very foundations of the great doctrine of Fluxions in the most unceremonious manner, brought down a mathematical storm on its author's head.

The scope of Berkeley's tract may be judged from its full title, which runs as follows: "The Analyst; or, a discourse addressed to an infidel Mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of Faith. 'First cast the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye.'" Although no names were mentioned, it is supposed that Halley was the "infidel Mathematician" glanced at. Of the apologetic aspect of the work we need only say that the piquant flank movement thus foreshadowed was a task well adapted to Berkeley's genius, and that he did not fail to make full use of each point maintained. His purely mathematical arguments are, however, so interesting and so little known in detail that no apology need be prefixed to a brief account of them.

We have at the beginning a clear and fair account of Newton's doctrine of Fluxions. "Lines are supposed to be generated by the motion of points, planes by the motion of lines, and solids by the motion of planes. And whereas

quantities generated in equal times are greater or lesser according to the greater or lesser velocity wherewith they increase and are generated, a method hath been found to determine quantities from the velocities of their generating motions. And such velocities are called fluxions ; and the quantities generated are called flowing quantities. These fluxions are said to be nearly as the increments of the flowing quantities, generated in the least equal particles of time ; and to be accurately in the first proportion of the nascent, or in the last of the evanescent increments. Sometimes, instead of velocities, the momentaneous increments or decrements of undetermined flowing quantities are considered, under the appellation of moments. By moments we are not to understand finite particles. These are said not to be moments, but quantities generated from moments, which last are only the nascent principles of finite quantities. It is said that the minutest errors are not to be neglected in mathematics : that the fluxions are celerities, not proportional to the finite increments, though ever so small ; but only to the moments or nascent increments, whereof the proportion alone, and not the magnitude, is considered."

After some ridicule of "successive fluxions,"—rather beside the point,—our author proceeds to attack a demonstration of Newton's which is certainly strong meat for the squeamish in logical matters, the proof of the fundamental rule for differentiating a product. It is to be found in Book II. Lemma II. of the *Principia*, and is as follows :

"Any rectangle, as AB, augmented by a perpetual flux, when as yet there wanted of the sides A and B half their moments, $\frac{1}{2}a$ and $\frac{1}{2}b$, was $A-\frac{1}{2}a$ into $B-\frac{1}{2}b$, or $AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$; but as soon as the sides A and B are augmented by the other half moments, the rectangle becomes $A+\frac{1}{2}a$ into $B+\frac{1}{2}b$, or $AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. From this rectangle subtract the former rectangle and there will remain the excess $aB+bA$. Therefore, with the whole increments a and b of the sides, the increment $aB+bA$ of the rectangle is generated. Q.E.D."

Berkeley objects that "the direct and true method to obtain the moment or increment of the rectangle AB" is to take $(A+a)(B+b)-AB$, which gives of course $aB+bA+ab$, in excess of the former answer by ab. "Nor will it avail to say that ab is a quantity exceeding small, since we are told that in mathematics the minutest errors are not to be neglected. Such reasoning as this for demonstration nothing but the obscurity of the subject could have encouraged or induced the great author of the fluxionary method to put upon his followers."

It must be admitted that the proof we have quoted exhibits the instinct rather of the mathematician than of the philosopher, or, as Berkeley uncompromisingly observed, "of which proceeding it must be owned the final cause or motive is obvious ; but it is not so obvious or easy to explain a just and legitimate reason for it, or show it to be geometrical." The answer which a modern mathematician might make, as to the *relative* smallness of ab, was not at that time so easy and plain ; in fact, a and b occupied very anomalous positions. They were "moments," or "momentaneous increments," or "the nascent principles of finite magnitudes," or the "velocities of increments." All these terms were used by Newton and were presumably intended to clear up the nature of a and b, but their effect on the critic was recorded thus : "Points which should be clear as first principles are puzzled ; and terms which should be steadily used are ambiguous." And again : "What are these same evanescent increments ? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities ? "

By his exposure of this neat way of glozing over an awkward point, Berkeley may be regarded as having at least found a chink in the giant's armour. Much encouraged, he goes on to the proof of the rule for differentiating x^n, which is to be found in the *Introduction* to the *Quadrature of Curves*. His remarks on this demonstration are of great importance.

Newton said, "Let the quantity x flow uniformly, and let it be proposed to find the fluxion of x^n.—In the same time that the quantity x, by flowing, becomes $x+o$, the quantity x^n will become $(x+o)^n$, that is, by the method of Infinite Series,

$$x^n + nox^{n-1} + \frac{n^2-n}{2} oox^{n-2} + \text{etc.}$$

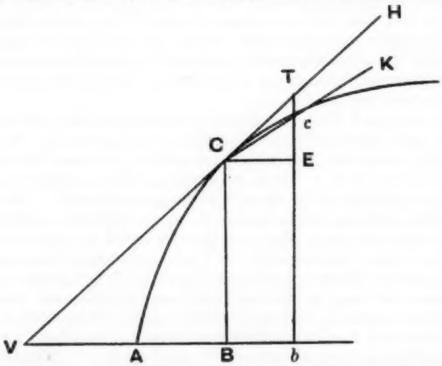
And the increments o and $nox^{n-1} + \frac{n^2-n}{2} oox^{n-2} + \text{etc.}$

are to one another as 1 and $nx^{n-1} + \frac{n^2-n}{2} ox^{n-2} + \text{etc.}$

Now let the increments vanish, and their ultimate ratio will be 1 to nx^{n-1}."

On this Berkeley commented, "It should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish, *i.e.* let the increments be nothing, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, *i.e.* an expression got by virtue thereof, is retained. . . . Hitherto I have supposed that x flows, that x hath a real increment, that o is something. And I have proceeded all along on that supposition, without which I should not have been able to have made so much as one single step. From that supposition it is that I get at the increment of x^n, that I am able to compare it with the increment of x, and that I find the proportion between the two increments. I now beg leave to make a new supposition contrary to the first, *i.e.* I will suppose that there is no increment of x or that o is nothing; which second supposition destroys my first and is inconsistent with it, and therefore with everything that supposeth it. I do nevertheless beg leave to retain nx^{n-1}, which is an expression obtained in virtue of my first supposition, which necessarily presupposed such supposition, and which could not be obtained without it."

A somewhat similar attack was made on a geometrical illustration which Newton had employed as a leading case in the *Introduction* to the *Quadrature of Curves.* Newton had said, "Let the ordinate BC advance from its place



into any new place bc. Complete the parallelogram $BCEb$, and draw the right line VTH touching the curve in C, and meeting the two lines bc and BA produced in T and V. Bb, Ec and Cc will be the increments now generated of the abscissa AB, the ordinate BC and the curve line ACc; and the sides of the triangle CET are in the *first ratio* of these increments considered as nascent, therefore the fluxions of AB, BC and AC are as the sides CE, ET and CT of the triangle CET, and may be expounded by these same sides, or,

which is the same thing, by the sides of the triangle *VBC*, which is similar to the triangle *CET*.

"It comes to the same purpose to take the fluxions in the *ultimate ratio* of the evanescent parts. Draw the right line *Cc*, and produce it to *K*. Let the ordinate *bc* return into its former place *BC*, and when the points *C* and *c* coalesce, the right line *CK* will coincide with the tangent *CH*, and the evanescent triangle *CEc* in its ultimate form will become similar to the triangle *CET*, and its evanescent sides *CE*, *Ec* and *Cc* will be *ultimately* among themselves as the sides *CE*, *ET* and *CT* of the other triangle *CET* are, and therefore the fluxions of the lines *AB*, *BC* and *AC* are in this same ratio. If the points *C* and *c* are distant from one another by any small distance, the right line *CK* will likewise be distant from the tangent *CH* by a small distance. That the right line *CK* may coincide with the tangent *CH*, and the ultimate ratios of the lines *CE*, *Ec* and *Cc* may be found, the points *C* and *c* ought to coalesce and exactly coincide. The very smallest errors in mathematical matters are not to be neglected."

The weakness of this plausible statement is clear. Berkeley comments, "It is particularly remarked and insisted upon by the great author, that the points *C* and *c* must not be distant one from another, by any the least interval whatsoever: but that in order to find the ultimate proportions of the lines *CE*, *Ec* and *Cc* (*i.e.* the proportions of the fluxions or velocities), expressed by the finite sides of the triangle *VBC*, the points *C* and *c* must be accurately coincident, *i.e.* one and the same. A point therefore is considered as a triangle, or a triangle is supposed to be formed in a point. Which to conceive seems quite impossible."

Although Berkeley considered that he had demolished the foundations of the Science of Fluxions, he did not altogether deny the utility of the method. "But then it must be remembered that in such case, although you may pass for an artist, computist, or analyst, yet you may not be justly esteemed a man of science and demonstration." And again: "In answer to this you will perhaps say, that the conclusions are accurately true, and that therefore the principles and methods from whence they are derived must be so too. . . . I say that in every other science men prove their conclusions by their principles, and not their principles by their conclusions. But if in yours you should allow yourselves this unnatural way of proceeding, the consequence would be that you must take up with Induction, and bid adieu to Demonstration."

The reader must be left to judge of the force of Berkeley's criticisms. To the compiler of this brief account they seem conclusive against any attempt to base the Differential Calculus on the idea of motion. Such attempts, however natural and useful, seem inevitably to lead to a logical quagmire, which the builder of an exact science must contemplate with dismay. Those who share this view will not deny to Berkeley, despite his lack of constructive ideas, some part of the praise which mathematicians of a later day have bestowed on the researches of Weierstrass.

W. D. Evans.

THE ACHIEVEMENTS OF GREAT BRITAIN IN THE REALM OF MATHEMATICS.*

I.

The inhabitants of the three kingdoms which constitute Great Britain did not come into contact with the civilisation of the ancient East. Nor, as far as we know, did they have the advantage of direct relations

* An address delivered to the International Congress of Historical Studies, London, April 4, 1913, by Prof. Gino Loria. Translated for the *Gazette* by kind permission of the Author.

with the Greeks—that race, beloved of Nature, which succeeded in placing all the sciences and all the fundamental arts on so secure a basis as to defy for two thousand years any attempt at radical innovation.

The Celtic invasion, which took place about 1000 B.C., entrusted to the Druids, the celebrated priests of the dominant religion, the task of preserving, diffusing, and extending the boundaries of knowledge, but of the substance, nature, and tendencies of the science of that period we are totally ignorant. Julius Caesar has left us the oldest description of the inhabitants that we possess, but he makes no reference to the intellectual life of the ancient Britons. A second Roman invasion about a century later, followed by a struggle extending over some fifty years, secured to Rome this remote addition to her Empire. Her occupation lasted for three centuries, during which the Britons enjoyed as some compensation for the loss of their independence a period of repose and of relative civilisation. In 410 A.D. the island was released from her allegiance. The departure of the Roman eagles exposed Britain to the cupidity of the pirates of North Germany, who made numerous more or less successful descents upon its shores between A.D. 350 and 450, and after a struggle which lasted two hundred years, completed the conquest of the country. The conquerors were unable to impose upon the native population their religion, their laws, or even their language. Under such circumstances it could not reasonably be expected that the Britons would be drawn by their unwelcome guests from the state of barbarism in which they were plunged.

A few gleams of light were thrown upon the scene by the visit of the band of missionaries sent in 597 by Pope Gregory the Great. They were the first to bring books, both sacred and profane, across the Channel. A few years later, the eloquence of St. Augustine, Archbishop of Canterbury, proved so effective that in the course of a century the last traces of Paganism had vanished from the land.

In the monasteries men found that repose and atmosphere which are congenial to the study of literature, and to quiet thought over the scientific problems that occupied the minds of that age. For a measure of the arithmetical skill of the time we look to the works of the Venerable Bede (672 or 673–735). The monk of Jarrow taught his 600 scholars in a monastery situated on the borders of England and what then was Scotland. His *De Computo vel Loquela Digitorum* and *De Ratione Unciarum* attest the intense desire of the "Father of English Learning" to communicate to every class of society all that the world had then to teach of the art of computation. England had begun her contribution to the literature of Mathematics.

II.

The example of Bede was followed by Alcuin (735–804). In his works, which in the main are historical and theological, the mathematician discovers with pleasurable anticipations a collection of problems in Arithmetic. The title is sufficient to indicate their immediate aim: *Propositiones ad acuendos Juvenes.*

But for many a long year no addition was made to this scanty body of English mathematical literature. It may be that the whole of the national energies were absorbed in the incessant struggles consequent on the inroads of the Danish and Norwegian pirates. Their incursions ceased only when the Norsemen sang their songs of victory over the native population. It was in 1066 that began the real history of the English race as the people of Great Britain. This far-reaching political revolution did not wrest from the hands of the clergy the monopoly

of learning and the control of public instruction, nor could it deprive the Latin tongue of its enviable prerogative of acting as the normal vehicle for the transmission of thought. On the other hand, the characteristic phrase, "An unlearned King is a crowned Ass"—attributed to Henry I., one of the early Norman kings—goes far to prove that culture, under the new régime, could look for aid and encouragement instead of oppression and contempt. In fact, it was not long after that the University of Cambridge came into existence, and schools and colleges were founded both there and at Oxford.

III.

The intellectual relations established in the age of the Crusades between the southern countries of Europe and ancient Greece, through Arabic channels, found an echo even in so remote a corner of the continent as that inhabited by the Anglo-Saxons. An English monk, Athelhard of Bath, translated for his fellow-countrymen the *Elements of Euclid* from the Arabic into Latin. Consider how fortunate it was that Great Britain should possess a work of such didactic value as "The Elements"! Think of the millions in Europe who never had this as their text-book, and draw up an approximate list of the innumerable forms of scientific research and criticism which might have been undertaken from such a starting-point. We are forced to rank Athelhard's translation as an event of the first order, not unworthy to be compared in the history of Mathematics with the appearance in 1202 of the *Liber Abaci* of Leonardo of Pisa (Fibonacci). It would be of the highest interest to know what reception was accorded by English minds to the messenger who brought from across the ages the greatest achievement of Greek thought, and to ascertain the precise period at which its beneficent influence is perceptible. But alas!

> These are questions none can answer;
> These are the problems none can solve.

That gifted genius Roger Bacon (1214-1294), "doctor mirabilis," who made so many important contributions to the progress of the various branches of Physics, left nothing (if we confine ourselves to his published works) which serves to throw any light on his statement that he held the exact sciences in such esteem that he did not hesitate to assert that Mathematics should be regarded as the Alphabet of all Philosophy. The same may be said with even greater emphasis of his colleague, John Peckham (1240–1292), and of the justly celebrated John Holywood—Johannes de Sacro Bosco—(?–1244 or 1256). The latter was one of the most eminent personalities in England in this century, and it was his constant care to extend throughout his native land such beneficent influence as he could reasonably be expected to exert.

IV.

Thomas Bradwardine (1290–1349), Archbishop of Canterbury, "the Profound Doctor," deserves a place in the history of Geometry for his works on perspective and on stellate polygons From his didactic works we can form a fair estimate of the quantity and quality of the Mathematics taught at the Anglo-Saxon Universities of the fourteenth century. If, as is possible, there are some of his writings lying buried in undeserved oblivion in some of the libraries of Great Britain, it is of the highest importance that they should be exhumed, examined by competent investigators, and published. The same remarks apply to the two Oxford Professors, Richard of Wallingford (about 1326) and

John Maudith (about 1340), whose works on Trigonometry are imperfectly known and deserve to be brought to the full light of day.

Such a search might very possibly bring to light other scientific writings now long forgotten, and thus fill the gap which we find at this stage in the history of English Mathematics. For while the two professors just mentioned belong to the first half of the fourteenth century, it is not till the last quarter of the fifteenth century that we find mention of a new mathematical personality. We allude to Cuthbert Tonstall (1474-1559), who, after pursuing the regular course at Cambridge and Oxford, proceeded to Padua. There he won a high and well-deserved reputation, distinguishing himself more particularly in Jurisprudence. But during his stay in Italy he was also deeply interested in mathematical investigations, and made a careful study of the works of Luca Paciuolo. So great, indeed, was his enthusiasm for our science, that before finally dedicating his life to political and religious affairs, to which he had decided to devote the whole of his energies, he bade a public farewell to the subject. This parting benediction took the form of a treatise mainly intended to set forth the principles of Algebra, then a new science, in which great strides had been made in "beautiful Italy" during the historic interval between Leonardo Fibonacci and Luca Paciuolo. Such was the origin of his *De Arte Supputandi Libri Quattuor* (1522), a work which was favourably received not only in his own country but by the whole of learned Europe.

V.

Tonstall may have been the first, but he certainly was not the only link in the scientific field between Italy and England. In fact, in the glorious Galilean epoch the intellectual relations between the two countries were both continuous and intimate. Antonio Favaro, who is thoroughly acquainted with everything that concerns the life, work, and school of Florentine Physics, gives * the names of Segeth, Southwell, Wedderburn, White, and Willoughby as those of men who spread far and wide the ideas and works of the greatest master that scientific Europe had yet produced, and made them known to their own countrymen. Another eminent geometer of that time, James Gregory (1638–1675) made several lengthy stays in Italy, as we know from letters of his which are still extant.† Two of his works were published abroad, one at Paris and the other at Venice.

Now, if we bear in mind that the methods of the Infinitesimal Calculus, in the elaboration of which Bonaventura Cavalieri and Evangelista Torricelli played so conspicuous a part, were conceived in that brilliant circle of which Galileo was the central figure, and if we remember that England was shortly to be hailed unanimously as the creator of the Calculus of Fluxions, we see at once arising the most interesting problem in the history of our science—the extent to which the genius of Newton was influenced by the work of Galileo. Such a piece of research undertaken without reference to nationality, and with the sole desire of reaching the truth, would certainly lead to most important conclusions, inasmuch as they must inevitably throw some ray of light upon the mysterious manner in which great and novel ideas reach their complete maturity.

* A. Favaro mentions the Englishmen of the School of Galileo in several chapters of his series, "Amici e correspondenti di Galilei" (published in the *Atti del R. Instituto Veneto di Scienze, Lettere ed Arti*), and in many places in his "Scampoli galileiani" (in the *Memorie della R. Accademia di Padova*).

† Cf. L. J. Rigaud, *Correspondence of Scientific Men of the 17th Century* (Oxford, 1841).

VI.

It would seem that the laudable efforts of Tonstall to avail himself of his personal experiences to diffuse a knowledge of the new methods and results of continental mathematicians were crowned with but moderate success. Not long after Robert Record (1510–1558), a court physician, could write of the English that in general intelligence they were superior to most, but that they were disgracefully ignorant of what others had done in the field of science. Determined to end this deplorable state of things as far as in him lay, he published an excellent treatise covering the whole ground of Arithmetic, to which he gave the significant title: *The Grounde of Artes.* It is difficult to say too much in its praise. It has a lasting place in mathematical literature as the first printed volume in which we find the sign of equality (=) which is still in use. So flattering was the reception of this book that the author later produced two others, dealing respectively with Algebra and Geometry. But a much greater influence upon his contemporaries was exercised by a famous mathematician of the reign of Queen Elizabeth—William Oughtred (1574–1660), a contemporary of Bacon and of Shakespeare, to whom we owe the *Clavis Mathematica* (in which are used for the first time the symbols × and : : for multiplication and proportion respectively). He also wrote an excellent treatise on Arithmetic, the long and well-deserved reputation of which is attested by the many editions and translations with which it was honoured, as well as by the fact that the author was encouraged to bring out an additional volume on the subject of Trigonometry.

With Record and Oughtred we approach the great stages of the epoch during which English mathematics passed from the period of infancy to that of fertile virility, in which the humble position of the scholar was changed for that of the master. The transition is marked by two names—John Napier (1550–1617) and Thomas Harriot (1560–1621).

The former travelled much in France, Germany, and Italy. His name is indissolubly connected with the logarithmic calculus which he, with the aid of his indefatigable compatriot Henry Briggs (1556–1631), placed at the disposal of the calculators of the world. Not content with the unexpected assistance he thus gave to those whose pursuits necessitated the carrying out of long and laborious computations, he applied his genius to what he called "local Arithmetic" (depending in principle on the expression of numbers in the binary scale), and taught in his *Rabdologia* an ingenious process of instrumental calculation.* And finally, it is well known to all that Spherical Trigonometry is indebted to Napier for certain rules and formulae which are still in daily use.

The place in the science to be accorded to Harriot has been a matter of some debate. Certain writers, basing their conclusions on his *Artis Analyticae Praxis* (1631), consider him as a rival of Vieta, while others regard him as a disciple, if not a plagiarist, of the great French mathematician. The question cannot be said to be settled one way or the other. The discussion was not entirely free from national prejudices, nor perhaps can a final answer be given until Harriot's manuscripts, which are religiously preserved in the Library of the British Museum, have been published, or, at any rate, closely examined. An old friend of mine, whose name is well known in the history of mathematics,†

* The apparatus invented by Napier is now exhibited in the South Kensington Museum.

† G. Vacca, "Sui manuscritta inediti di Thomas Harriot" (*Bolletino di bibl. e storia delle scienze matematiche*, t. v. 1902, pp. 1-6).

examined these manuscripts some ten years ago. The results of his scrutiny were such as to encourage the belief that a more complete and leisurely investigation would be productive of important results. For the moment it is sufficient to add that Harriot introduced the name "canonical equation," and the symbols still in use ($>$, $<$) for "greater than" and "less than."

VII.

Thus far we have confined ourselves to number and measurement—no mention has been made of figured extension. But, while English work on the former is characterised by a real originality, that on the latter assumes in the main the form of mere comments on philological and critical investigations. Of the first type are the lectures delivered at Oxford by Sir Henry Savile (1549–1622), the munificent founder of the two mathematical chairs in the university, and those delivered at a later period at Cambridge by Isaac Barrow (1630–1677), the master, friend, and precursor of the greatest genius England has given to the exact sciences.

We now approach the excellent editions of Archimedes, Aristarchus, and Ptolemy which we owe to the labours of the celebrated John Wallis (1616–1703), one of the most active and original investigators known to history, and the only calculating prodigy who has ever made any contribution of real value to mathematical knowledge.*

In all probability it is to his unwearied study of the scientific writings of classical antiquity that we must trace the inspiration and power displayed by Wallis in the composition of his famous *Arithmetica Infinitorum*—a most admirable preparation for the new calculi which were shortly to come into existence. In it we find for the first time the familiar symbol, ∞, to denote "infinity," and here, too, the student of analysis will find Wallis's most elegant expression for π as a product of infinite factors. To Wallis also belongs the credit of the discovery of the first algebraical surface not of revolution of a degree higher than two (I allude to the celebrated cuno-cuneus or circular wedge). Nor must it be forgotten that he shares with Lord Brouncker the honour of investigating the theory of continued fractions, and that he saved from irreparable loss John Caswell's excellent treatise on Trigonometry. He had a considerable share in the founding of the Royal Society of London, and saw that justice was done to his pupil William Neil (1637–1670), who had discovered the first curve which was algebraically rectified (the semi-cubical parabola). Christopher Wren, the famous architect of St. Paul's Cathedral, followed with the rectification of the ordinary cycloid.

But while work of this kind was suited to the genius of Wallis, he also, unlike many of his contemporaries, experienced and manifested on many occasions a keen interest in the historical evolution of scientific ideas. In his *Treatise on Algebra, both Theoretical and Historical* he showed that he could not always keep under control his predilections for his own countrymen and his dislike to foreigners, so that many of his assertions were challenged, and many of his conclusions were exposed to objections of considerable force. Nevertheless, there remains a serious attempt to blend into one organic whole the narrative of the struggle and the display of the trophies of victory. But there is ample justification for the remark, that Wallis found in his own country

* [William Rowan Hamilton at the age of twelve was on more than one occasion a match for Zerah Colburn (v. *Encyclopaedia Britannica*, "Hamilton, W. R.") W. J. G.]

timid admirers and few followers. We must not forget or pass over in silence the fact that Wallis flourished in a period that was marked by an active and peaceful exchange of ideas across the Channel. This is most marked in the development of the Theory of Numbers, striking evidence of which exists in the correspondence which exists between Pierre Fermat, that French mathematician of unique power, and Sir Kenelm Digby (1603–1665). And the name of John Pell (1610–1685), given by Euler, though somewhat inappropriately, to the fundamental equation in indeterminate analysis of the second degree, is a further proof of the collaboration that went on at that time between the mathematicians of England and the Continent, with the object of overcoming the obstacles to the solution of the important problems that challenged the skill of the mathematical world of that period.

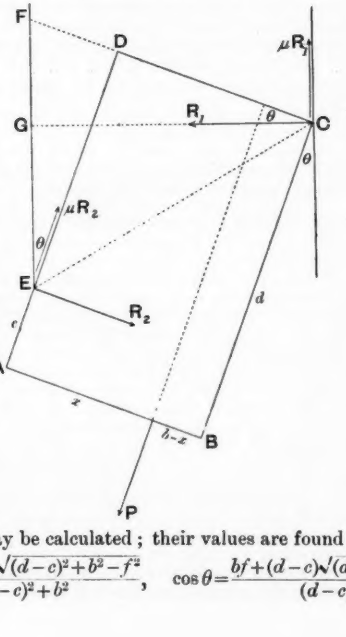
(*To be continued.*)

ANSWERS TO QUERIES.

[81, p. 341, vol. vi.] The diagram gives an exaggerated picture of the state of affairs in the most general case, viz. (i) when the drawer is not tightly-fitting, and (ii) when it has been pulled out some distance before jamming. The forces acting are as shown.

$AB=b$; $BC=d$. Let the force P be applied at a point in AB distant x from A $(x \neq b/2)$, and let $AE=c$. Then $ED=d-c$, $EC=\sqrt{(d-c)^2+b^2}$, and if the distance CG between the slides of the drawer be f,

$$EG=\sqrt{(d-c)^2+b^2-f^2}.$$



Then $\sin\theta$, $\cos\theta$ may be calculated; their values are found to be

$$\sin\theta=\frac{f(d-c)-b\sqrt{(d-c)^2+b^2-f^2}}{(d-c)^2+b^2},\quad \cos\theta=\frac{bf+(d-c)\sqrt{(d-c)^2+b^2-f^2}}{(d-c)^2+b^2}.$$

Taking moments about E and C, and resolving all the forces in the direction of P, we have the three equations,

$$Px = R_1\sqrt{(d-c)^2+b^2-f^2}+\mu R_1 f, \quad \text{.........(1)}$$

$$P(b-x)=\mu R_2 b - R_2(d-c), \quad \text{.........(2)}$$

$$P=\mu R_2 + R_1(\mu\cos\theta-\sin\theta), \quad \text{.........(3)}$$

and, eliminating R_1, R_2 between these equations, we get

$$P=\mu\frac{P(b-x)}{\mu b-(d-c)}+(\mu\cos\theta-\sin\theta)\frac{Px}{\mu f+\sqrt{(d-c)^2+b^2-f^2}}, \quad \text{.........(4)}$$

which is the required condition. On substitution of the values of $\sin\theta$, $\cos\theta$ and reduction, this becomes:

$$x\{Ab-f(d-c)\}\mu^2-\{2bfx-b^2f-f(d-c)^2+2Ax(d-c)\}\mu$$
$$=\{Abx-Ab^2-fx(d-c)-A(d-c)^2\}, \quad \text{.........(5)}$$

where $$A\equiv\sqrt{(d-c)^2+b^2-f^2}.$$

Special cases:

I. If the drawer fits well, $f \doteqdot b$, $\sin\theta \doteqdot 0$ and $\cos\theta \doteqdot 1$, and equation (4) becomes:

$$1=\frac{\mu(b-x)}{\mu b-(d-c)}+\frac{\mu x}{\mu b+(d-c)}$$

(taking the positive square root),

and this reduces to $$\mu=\frac{d-c}{2x-b}. \quad \text{.........(6)}$$

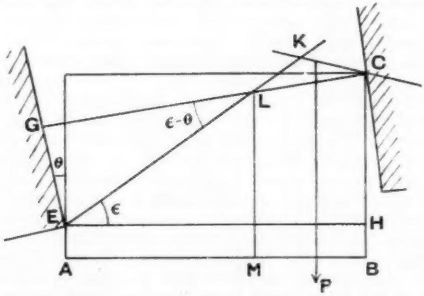
This equation shows (i) that the greater x is, the less is the coefficient of friction necessary for jamming, (ii) that there is more chance of jamming after the drawer has been pulled out some way than initially, and (iii) that, since $2x-b\equiv m$, twice the distance of the point of application of P from the mid-point of AB, no force however great will pull the drawer out unless its length $d>c+\mu m$. [Cf. Loney, *Statics*, p. 327, Ex. 47.]

II. By putting $c=0$, we get the condition for jamming before the drawer moves at all, viz.

$$\mu=\frac{d}{2x-b}.$$

JOHN M'WHAN.

[81, p. 341, vol. vi.] The problem can be treated graphically. Taking CG normal to the guide at the point C and EH normal to the side of the drawer at E, if we draw CK to make $G\hat{C}K=\epsilon$, the angle of friction, and EK from E



making $H\hat{E}K=\epsilon$ and cutting CG in L, CC in K, then we form a triangle CLK, outside of which the reactions at C and E cannot meet.

Hence, in order that the drawer may jam, the line of action of P must pass to the left of P, and if its direction is $\perp^r$ to AB, its point of application must be between M and B, where LM is the perp. from L on AB.

Take $$CH=a \quad [=d-c].$$

Now $$MB=CL\cos\theta,$$

and
$$\begin{aligned} CL=GC-GL&=b\cos\theta+a\sin\theta-GE\cot(\epsilon-\theta)\\ &=b\cos\theta+a\sin\theta-(a\cos\theta-b\sin\theta)\cot\overline{\epsilon-\theta}\\ &=\frac{b\sin\epsilon-a\cos\epsilon}{\sin(\epsilon-\theta)};\end{aligned}$$
$$\therefore\ MB=(b\sin\epsilon-a\cos\epsilon)\frac{\cos\theta}{\sin(\epsilon-\theta)}.$$

If we put $\theta=0$, this becomes $MB=b-a\cot\epsilon=b-\dfrac{a}{\mu}$ or $AM=\dfrac{a}{\mu}$.

This differs from Mr. M'Whan's solution, where it seems to be assumed that, for jamming, the max. friction must be in action both at C and at E. This would imply that the line of action of P should pass through K.

R. F. M.

REVIEWS.

Elements of Algebra. Part I. By G. St. L. Carson and D. E. Smith. Pp. 346 (with Tables). 3s. 1914. (Ginn & Co.)

The modern principles that govern the teaching of elementary Algebra are now so well understood that it is really only the author's skill in the presentation of the subject and his power to invent interesting and instructive examples that are called into play. The insertion of really new fundamental methods is not to be expected—at all events until the ways and means now in vogue have had a fair trial.

The present book is Part I. of a treatise on School Algebra, and must be regarded as an authoritative work on the subject. It consists of some 350 pages, and deals in eleven sections with the following subjects: Introduction to Algebra, Positive and Negative Numbers, Graphs, Addition and Subtraction, Multiplication and Division, Simple Equations, Simultaneous Equations, Special Products, Factors, Fractions, Logarithms. Besides these tutorial sections, there are Revision Papers, Miscellaneous Examples, History of Algebra, Tables, Index.

The book proceeds on the thoroughly sound pedagogic principle that the pupil ought to have a good concrete knowledge of Arithmetic and its processes before he attempts Algebra. This being assumed, the book generalises the arithmetical notions already in the pupil's mind, and shows how the use of formulae arises and how a formula is a convenient short-hand, and saves repetition and trouble. Thus the pupil's sympathy is enlisted from the start. By the wholesale use of concrete numerical examples, it is shown how the laws of Algebra were invented to meet the needs of Practical Life. Anything of the nature of abstract philosophy is studiously avoided. The teacher will find the section on Graphs fresh and stimulating reading, and it will prove an instructive part of the book if he leads his pupil carefully through it himself, but it will prove a path dangerous and full of pitfalls if he allows his pupils to traverse it without a guide. Many of the sets of examples are varied in their subject matter, and the pupils will like the topics wherewith they deal. So far the book is excellent.

The chief drawback of the book lies in the fact that 350 pages are required to cover so little ground. Furthermore, the applications do not follow hard after the algebraic drill. Throughout the book there are long chains of examples, all of the same pattern, and the boy sees no applications ahead and will not be able to apprehend what it is all coming to. He will find these sections very dull, and his faith in their usefulness will be severely taxed. Again, in some of the sections of examples which are intended to be problematic, the range of subjects is very

narrow, and the pupil will get weary with this monotony. We need only quote page 129. Also, most teachers will probably agree that simultaneous equations of three and also four unknowns are out of place in an elementary book.

On the whole, the book errs probably on the side of being rather solid and unwieldy for young boys, but teachers are of all sorts, and some will like it and some will not. It is certainly not one of those books concerning which there will be undivided opinion as to its practical utility in the class-room.

William P. Milne.

[We have also received the following appreciative comments from a valued contributor.]

Nothing but praise can be given to the manner in which the authors of this book have accomplished the task they have set themselves, that of providing an introduction to Algebra, striking the happy mean between the methods of "the ultra-conservatives and the extreme iconoclasts." The whole work is full of ideas, valuable alike to student and teacher, set out with a wealth of illustration, and strengthened with a profusion of examples carefully graded.

But the expressed idea is that Algebra should be developed slowly and carefully from Arithmetic by means of the "formula": the equation is looked upon as the outcome of the formula. I consider this is a rather dangerous idea to propound. One of the first things a beginner should appreciate is that the formula is an equation of sorts.

Lest I should be deemed guilty of egoism, in thus attacking such well-known mathematicians as the authors on points of presentment of their subject, I wish to explain that the following remarks are an expression of the result of 20 years' experience in teaching the elements of Mathematics; five years in a boys' school with the lower forms, and fifteen years "pure" and "practical" mathematics to that beginner, who is the hardest of any to teach, the adult technical student. I have experimented largely with different methods of presentment, judging the success of a method not at all on any examination of the year's work, but on the ease with which the students assimilated the work of the second year. Hence, what follows is a frank expression of opinion founded on this experience, and not an attempt to theorise dogmatically on the subject of the teaching of Elementary Mathematics.

I have found that the student gets hopelessly confused in the later stage, if in his elementary work he is shown on the blackboard that the formula for the area of a rectangle can be written either as

$$\text{length} \times \text{width} \quad \text{or} \quad l \times w.$$

He gets the idea that l and w stand for "so many inches" or "so many feet," *i.e.* quantities and not numbers. The authors state in their preface that success in Algebra depends on a thorough grasp of the elements of Arithmetic; and that the latter, therefore, should not be commenced until easy fractions and simple Mensuration have been mastered. If the student's ideas are confused, these essentials never will be mastered. The relatively easy idea to grasp, but immensely difficult idea to master,* that of the simplest fraction, should be deferred, literal numbers being freely used with simple mensuration as soon as the student has learnt his first four rules. This, I take it, follows the historical method in which pebbles, indivisible into parts, were used for calculation. The student is not doing Algebra when he uses the formula $A = l \times b$, in which l and b stand for integers; but he is doing Algebra when he says that $6 \times 1\frac{1}{2} = 9$.

To master his Arithmetic he must be chained down to one easy idea at a time. The oft-repeated phrase "From the concrete to the abstract" is useless in the majority of cases. "From the particular to the general" is the only motto that will ever give satisfaction in all cases. For instance, what boy of (say) eleven, let alone a technical student, would ever get a clear idea of multiplication by a negative number from such an explanation (an explanation from which the rule is deduced), as is given on page 66?

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* I quote from Prof. Smith's *Teaching of Arithmetic*, p. 7, "We are impatient that a child stumbles over common fractions, and yet so difficult did the world find the subject, that for thousands of years only the unit fraction was used."

"If a man save £2 each month, 4 months ago (-4 from now) he was £8 worse off than now.

That is $$(+2)\times(-4)=(-8)."$$

Very few beginners could, even with the help of the teacher, express the first two lines symbolically as the third; much less could they understand that it was given as a "proof" or, say, a reason for a rule, especially, when they found that it was very hard for them to find any other illustration of a negative multiplier than a negative "time."

Further, the authors are bound to state that "the same reasoning evidently holds when several numbers are multiplied together." What concrete illustration can be given of the product

$$(+2)\times(-3)\times(-5)=(-6)\times(-5)=(+30)?$$

The only confidence that the student gains from the method of explaining is a confidence in the teacher being able to "explain" any statement, no matter how unreasonable it may appear.

Again, the use of a scale or squared paper to provide what amounts to a "proof" gives the enquiring boy occasion to ask why you take points at *equal* distances along the scale to represent the numbers of the natural scale.

I believe I am correct in saying that 90 per cent. of boys like arithmetic, perhaps for the fact that there is only one answer, and they either get the correct answer to a question or they do not. This liking for dealing with figures should be taken advantage of to the utmost extent. Generalise his elementary notions of the fundamental operations so as to include negatives, fractions and surds, on the understanding that it would be inconvenient to have any laws governing operations other than those which can be proved for operations with positive integers. This can easily be done without the harder formal work of elementary number theory. Introduce at every point suitable illustration of the use of the new classes of numbers, such as abound in the book under criticism, but do not attempt to "prove" or give "reasons" for, rules of pure number from a consideration of quantities. I have tried the above suggested method for the last few years with students in First Stage "pure" and "practical" Mathematics, and find that, though apparently there is very slow progress in the first year, I am repaid doubly in the second year's work. As a good example, no student can grasp logarithms in the second year when he looks on them as generalised indices, unless he has been shown that a meaning can be assigned to such a symbol as $10^{0\cdot3010300\ldots}$. The pity of it is that this can be shown in such an easy way; cf. pp. 8, 9 of Prof. Bowley's *General Course in Pure Mathematics.*

However, taking for granted that Algebra will still be taught from the text, "From concrete to abstract," teachers can have no exposition anywhere approaching this book. Whether it will succeed as a class text-book is a matter for the future to decide.

It will not be the text-book solely that is on its trial, it will be the method as well.

The School Algebra. By A. G. Cracknell. Pp. 568 + lxxvii. Price 5s. 1914. (Clive & Co.)

The Mathematical Works published by the University Tutorial Press are noteworthy for the thoroughness of their treatment—they leave no detail untouched and no difficulty without explanation. Mr. Cracknell's *School Algebra* is no exception to this rule, and the boy of thirteen who begins Algebra with this text-book of nearly 600 pages to digest before he can use the Binomial Theorem may well be filled with dismay.

The writer of a text-book on Algebra for schools is, however, in a very difficult position. He must try to bring a subject which is based on difficult conceptions to the level of a very immature brain, and is meanwhile at the mercy of the mathematical expert who is ready to pounce on any incorrect presentation of the theory of negative operators.

Mr. Cracknell attempts a compromise with a fair amount of success, but he seems to have been hampered by a feeling that examination candidates must be able to reproduce proofs of theory, and yet he is indisposed to provide them with a rigorous method of treatment.

Most schoolmasters in practice are content at first to illustrate algebraic laws and to show that they are reasonable—the consideration of the proofs as to the generality of algebraic operations is left to a later stage, when the student is more mature.

Mr. Cracknell defines the negative sign as implying both subtraction and "reversal"; he postpones to Chapter XIV. the explanation of the Laws of Signs with reference to negative "quantities," but there states that no general proof can be given for Multiplication and Division.

The treatment of graphs and gradients is excellent, but the chapter on Variation would be improved if some of the graphical work had been included as definite illustrations of variation.

Besides a very large number of examples in the text, there are at intervals collections of excellent miscellaneous examples, numbering nearly 500.

The book has been compiled with great labour and care; it is not by any means of a revolutionary type nor is it, indeed, particularly original It will probably be welcomed by many teachers who fight shy of some of the more modern text-books, but who require some advance on the older works which are becoming out of date.

R. C. F.

The "Conway" Manual of Navigation: being a complete summary of all Problems in Navigation and Nautical Astronomy. By J. Morgan, T. P. Marchant and A. L. Wood. Pp. 79. Price 5s. 1914. (J. D. Potter, London.)

The main purpose of this work is adequately expressed in the preface as "a collection of formulas and methods required for solving plane and spherical triangles." In addition, some of the more special problems of Navigation are completely worked out, such as: latitude by meridian altitude of the sun; latitude by altitude of *Polaris*; longitude by altitude of sun or star, out of the meridian. Two fully worked-out examples of the so-called New Navigation complete the volume. These are the determination of the ship's position, (1) by two observations of the sun and the run between the observations; (2) by the simultaneous observation of two stars. The former of these is perhaps the commonest method employed by modern navigators for finding their position, and it is the one which is taught to cadets before they leave the R.N. College, Dartmouth.

As a compendium of formulas the work is admirable, and consequently as a handy book of reference its value is very great. The authors disclaim any pretensions to having written a treatise on Navigation, and some of the criticisms given below may in consequence appear to be wanting in fairness.

The proofs are set out without any waste of words and with the barest minimum of explanation. Some would be unintelligible to a learner without a great deal of assistance from a teacher. The usual proof of the fundamental cosine formula in spherical trigonometry from the cosine formula in plane trigonometry is given. The sine formula is deduced from the cosine formula by showing that $\dfrac{\sin A}{\sin a}$ is a symmetrical function of a, b and c, just as we might prove the sine formula in plane trigonometry by showing that

$$\frac{\sin A}{a}=\frac{\text{twice the area of the triangle}}{abc}.$$

Apart from the fact that an elegant geometrical proof is available, a proof of this kind seems quite out of place in a practical text-book. The geometrical proof is short and instructive; in addition, it goes back to first principles and definitions.

As already mentioned, two problems are worked out by the method of the New Navigation. This method is a development and improvement of the Sumner's Line, which has been used for nearly a century by navigators, and is due to Marcq St. Hilaire, who published an account of it in 1880. The time-honoured independent observations for latitude and longitude had therefore been for many years discarded before St. Hilaire's method was adopted. The principle of St. Hilaire's method is the comparison of the *observed* altitude of a heavenly body with the *calculated* altitude obtained from the assumed dead reckoning position of the ship. This enables the navigator to put down on the chart a position line, on which the ship must be. A further similar observation when the heavenly body's bearing has changed some 25° is made, and another position line is obtained which by

its intersection with the previous position line gives the ship's position. This mode of procedure enables all observations to be similarly dealt with, and the process is greatly facilitated by tables which give the azimuth or true bearing of the heavenly body. St. Hilaire's process may be readily combined with chart-work. Since in these days all dead reckoning is done graphically on charts, a position can frequently be fixed by a combination of a position line obtained from an observation with a position line obtained from a land fall or observation of a known landmark.

The amount of information condensed into the eighty pages of this work is remarkable. Each problem is worked out on a separate page, so that the whole scheme may be seen at a glance. The setting-out of the work in some definite, orderly system is insisted on, for it enables the computer to check his work readily at all stages. The examples are worked out for the *Nautical Almanac* of 1910. The *Nautical Almanac* of 1914 gives the declination of the sun and the equation of time for every two hours of the day, so that all interpolation can be done mentally. A great deal of the interpolating work in this manual is therefore now quite unnecessary.

The name of the publisher, Mr. J. D. Potter, the well-known agent for Admiralty charts, is a sufficient guarantee of the excellence of the figuring, printing and diagrams.

Matriculation Mechanics. By William Briggs and G. H. Bryan. Third Edition. Pp. viii + 363. Price 3s. 6d. 1914. (University Tutorial Press, Ltd.)

One cannot doubt from the success of this work that it serves admirably as a text-book for the London Matriculation Examination. At the same time it must be confessed that in some respects it is much behind the times. In their preface the authors state that one of their reasons for treating Statics without Trigonometry is because "the solution of most illustrative problems involving angles depends on the properties of *two* particular triangles." These two particular triangles are the ones with which we are familiar in the forms of the 45° and 60° set squares. Why illustrative problems should involve these angles we are not told. Nothing has tended more to make the subject of Mechanics unreal to beginners than the artificial importance given to the angles 30°, 45° and 60°. R. M. M.

Transcendenz von e und π. By Gerhard Hessenberg. Pp. 106. 3 marks. 1912. (Teubner.)

The object of this book is to explain how the coping stone was placed, within the lifetime of most of us, upon a mathematical structure which had been slowly rising for four thousand years, during which more than a hundred generations endured the lot of mankind.

Though familiar to all readers of such works as Klein's *Leçons sur certaines questions de Géométrie élémentaire*, Vahlen's *Constructionen und Approximationen*, Dr. Glaisher's articles in Vols. 2 and 3 of the *Messenger*, or Herr Rudio's Archimedes, Huygens, Lambert, Legendre, the following brief statement of the history may be worth giving:

1. In an Egyptian papyrus entitled "The Book of the Knowledge of all Dark Things," Ahmes states that the area of a circle is equal to that of the square on nine-tenths of the diameter ($\pi = 3{\cdot}1604$). Taking his date to be approximately 2000 B.C., some seventeen centuries elapsed before the second great step was taken by Archimedes.

2. Archimedes assumed (as is well known) that the circumference of a circle was intermediate between that of a regular circumscribed and a regular inscribed polygon of n sides. He employed Euclidean methods to show that, starting with the regular inscribed hexagon, the perimeters of inscribed and circumscribed polygons of 12, 24, 48, ... sides could be calculated by extraction of square roots only. [$3\frac{10}{71} < \pi < 3\frac{10}{70}$.]

3. Yet another nineteen centuries elapsed before the Arabic notation enabled the method of Archimedes to be extended by Vieta (10 places of decimals), Adrian Romanus (15), and Ludolph van Ceulen (35).

4. Now followed, within a few years, the fourth great advance. The discovery of the calculus led to convenient series, whose evaluation required the operation

of arithmetical division only instead of that of extraction of the Archimedean square roots. The effect is at once visible in the 100 decimal places obtained by Machin in 1706. This particular line of advance ends with the 707 decimal places given by Shanks (1873).

5. Up to this point the circle squarer had two possibilities open to him. Firstly, though it was now certain that no simple value of π such as $3\frac{1}{7}$ or $3\frac{1}{8}$ could be correct, it might still be hoped, with a waning hope, that after a few more places the calculation would "come out" and π prove to be rational. Secondly, the case of a square and its diagonal showed that it might be possible to give an accurate geometrical construction for π as well as for $\sqrt{2}$, even though π were arithmetically irrational.

The fifth advance was due in chief measure to Lambert, who, in 1766, published in the form of "preliminary hints for circle squarers," a proof that π cannot possibly be *rational*. Henceforth the only legitimate hope for the circle squarer lay in the possibility of a geometrical solution.

6. Now comes an apparently unconnected movement, for although Euler had noticed the remarkable equation $e^{\iota\pi}+1=0$, and attached, it is said, some mystical importance to the conjunction of e, ι and π, no one suspected that the key of the puzzle depended on this equation.

In 1815 Fourier proved that e could not be rational; Liouville in 1844 (*Comptes Rendus*, xviii. pp. 833, 910) showed that e could not be the root of a quadratic equation with rational coefficients.

The proof in outline was, that if a, b, c are rational and hence, without loss of generality, integral, we must have, say $ae+c/e=b$, and therefore

$$a\left(1+1+\frac{1}{2!}\cdots\right)$$
$$+c\left(1-1+\frac{1}{2!}\cdots\right)=b.$$

Multiplying up by $(n-1)!$, and discarding integers,

$$a\times \text{a fraction}+(-1)^n c\times \text{a fraction must be zero.}$$

By proper choice of n, a may have the same sign as $(-1)^n c$, and thus the sum of two small positive fractions will be zero, which is absurd.

In 1873 Hermite gave a proof, resting on the same general idea, that e cannot be a root of a rational algebraic equation with integral coefficients.

In 1882 Lindemann extended the proposition to the case of an equation with algebraic coefficients and exponents.

Since $e^{\iota\pi}+1=0$ and e cannot satisfy an algebraic equation, it follows that π cannot be an algebraic number.

Thus the question of ages was ended. Not only is it impossible for a Euclidean construction with rules and compasses to "square the circle," but it is useless to call in aid a conic or cubic or n-ic.

Since 1882 many mathematicians have laboured to simplify the reasoning of Lindemann's proof. The difficulty is in the start. Herr Hessenberg drily remarks that lectures on advanced mathematics remind him of nothing so much as of mountaineering in a dense fog when, without a trusty guide, to wit, the lecturer's MS., the way would certainly be lost.

The general idea, to show that if e satisfies an algebraic equation, then we shall be driven to conclude that an integer can equal a proper fraction is simple enough, but the mathematical operations by which the details are carried out are complicated.

Herr Hessenberg's incidental comments are always amusing and interesting, and he has attained a considerable measure of success in his attempt to lead up to the notions by which the proof was developed. A very clear account will also be found in Vahlen, *Constructionen und Approximationen*, recently reviewed in the *Gazette*. [Since the above was written, Professor Hobson's "Squaring the Circle" has appeared, in which a further simplification of the proof has been effected.]

C. S. JACKSON.

The Algebra of Logic. By L. COUTURAT. Authorised English Translation by L. G. Robinson. Preface by P. E. B. Jourdain. Pp. xv+97. $1.50. 1914. (Open Court Co.)

Miss Robinson may be congratulated on her excellent translation of a work which will be welcome to many in its English form. To those who are interested in the subject, the book is too well known to need more than this passing reference. There are few readers who will not appreciate the interesting historical retrospect given by Mr. Jourdain in his preface, and agree with him that M. Couturat shows "in an admirably succinct form, the beauty, symmetry and simplicity of the calculus of logic regarded as an algebra."

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By H. CRABTREE. Second Edition. Pp. xv+193. 7s. 6d. net. 1914. (Longmans, Green.)

The second edition of Mr. Crabtree's well-known introduction to the elementary theory of gyroscopic phenomena is enriched by about fifty pages of additional matter, mainly in the form of appendices. These deal with the swerve of the "sliced" golf ball, the drifting of projectiles, and the behaviour of a spinning top under various conditions. The gyro-compass is described in Chap. V., and a fuller account of the theory and equations of motion of Anschütz's invention is given in the last appendix. Two plates have been added, showing respectively Schilowsky's Monorail Car and the "damping" device in the Gyro-Compass.

CORRESPONDENCE.

THE EDITOR OF THE *Mathematical Gazette.*

DEAR SIR,—In a note on Desargues' Theorem in the October number of the *Mathematical Gazette*, Dr. D. M. Y. Sommerville discusses an "interesting representation," by Major Dixon, "of a plane geometry in which straight lines are represented by closed curves on a closed convex surface devoid of singularities." In a discussion on such a system, I do not desire to intervene, but I cannot allow the following footnote at the bottom of page 394 to pass unchallenged:

> "Essentially the same form of proof is given by J. L. S. Hatton, *Projective Geometry* (Camb. Univ. Press, 1913), p. 19. The elegance of the proof disguises its logical unsoundness."

There is, I submit, no illogical unsoundness in the proof as used in my book. It depends on the following facts:

(1) That given two points A and B on a straight line and the ratio $\frac{AP}{BP}$ (sign being taken into account), the point P on the straight line is uniquely determined;

(2) That, defining $(ABCP)$ as $\frac{AC}{BC}:\frac{AP}{BP}$, if A, B, C are given and also the value of $(ABCP)$, then the point P is uniquely determined on the straight line ABC, provided A, B, C are collinear.

(3) That in a real projection $(ABCP)$ is unaltered.

(4) Hence by (2) and (3), if $(ABCD)=(AB'C'D')$, then the straight lines BB', CC', DD' are concurrent.

Nothing, I submit, could be more simple and straightforward.

As far as I can follow Dr. Sommerville's argument, the fact that (4) is a particular case of the "Fundamental Theorem of Projective Geometry" renders this proof such that "the elegance of the proof disguises its logical unsoundness." I have yet to be convinced that every theorem must be stated and proved in the first instance in its most general form, nor am I at present prepared to grant that there is any logical unsoundness in proving and using the Binomial Theorem for a positive integral index before it has been proved for a complex index.

If such a simple, straightforward and logical proof as that under discussion can be a matter of any interest, I may say that I have given it in my lectures for about fifteen years, and that about ten years ago one of my students drew my attention to the same proof in a small German text-book by Dr. Doehlemann.—I am, yours very truly, J. L. S. HATTON.

East London College (University of London),
2nd November, 1914.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian acknowledges with thanks the gift of 39 volumes of valuable mathematical works from the Library of King's College for Women.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 or 2 „ „ No. 8.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

ERRATUM.

Vol. vii. p. 87, line 4, *to* Feb. 8 *add* 1913.

BOOKS, ETC., RECEIVED.

Lehrbuch der Differential- und Integralrechnung. By J. A. SERRET. Fourth and Fifth editions. Edited by G. SCHEFFERS. Vol. III. *Differentialgleichungen und Variationsrechnung.* Pp. xiv + 735; 13 m., bound 14 m. 1914. (Teubner.)

A New Analysis of Plane Geometry, Finite and Differential. By A. W. H. Thompson. Pp. xvi + 120. 7s. net. 1914. (Cambridge University Press.)

Problèmes d'Arithmétique Amusante. By P. DELENS. Pp. viii + 164. 2 fr. net. 1914. (Vuibert, Paris.)

The School Algebra. By A. G. CRACKNELL. Pp. viii + 568 + lxxvii. 5s. With or without answers. 1914. (University Tutorial Press.)

Time as a Fourth Dimension. By Prof. R. C. ARCHIBALD. Pp. 4. Reprint from the *Bull. of the Amer. Math. Soc.* Vol. XX. Pp. 409-412.

The American Journal of Mathematics. Edited by FRANK MORLEY. Vol. XXXVI. No. 3. July 1914. 5$ per ann. (The Johns Hopkins Press, Baltimore, Md., U.S.A.)

On a certain completely integrable System of Linear Partial Differential Equations. E. J. WILCZYNSKI. *On the Connection of an Abstract Set, with Applications to the Theory of Functions of a General Variable.* A. D. PITCHER. *On Series of Iterated Linear Fractional Functions.* R. D. CARMICHAEL. *The Derivative of a Function of a Surface.* C. A. FISCHER. *Some Invariants and Covariants of Ternary Collineations.* H. B. PHILLIPS. *A Geometrical Application of the Theory of the Binary Quantic.* F. P. LEWIS.

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXVI. No. 4. Oct. 1914. 5$ per ann. (Johns Hopkins Press, Baltimore.)

The Quartic Curve and its Inscribed Configurations. H. BATEMAN. *On the Continuity of a Lebesgue Integral with respect to a Parameter.* J. K. LAMOND. *Geometry on Ruled Surfaces.* S. LE LEFSCHETZ. *Restricted Systems of Equations.* (Second Paper.) A. B. COBLE. *Character of the Solutions of certain Functional Equations.* T. E. MASON. *Binary Conditions for Double and Triple Points on a Cubic.* L. A. HOWLAND. *Modular Invariants of Two Pairs of Cogredient Variables.* W. C. KRATHWOHL.

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